A new class $\hat{o}_N$ of statistical models: Transfer matrix eigenstates, chain Hamiltonians, factorizable S-matrix

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Abstract

Statistical models corresponding to a new class of braid matrices ($\hat{o}_N$; $N \geq 3$) presented in a previous paper are studied. Indices labeling states spanning the N^r dimensional base space of $T^{(r)}(\theta)$, the r-th order transfer matrix are so chosen that the operators W (the sum of the state labels) and (CP) (the circular permutation of state labels) commute with $T^{(r)}(\theta)$. This drastically simplifies the construction of eigenstates, reducing it to solutions of relatively small number of simultaneous linear equations. Roots of unity play a crucial role. Thus for diagonalizing the 81 dimensional space for $N = 3$, $r = 4$, one has to solve a maximal set of 5 linear equations. A supplementary symmetry relates invariant subspaces pairwise ($W = (r, Nr)$ and so on) so that only one of each pair needs study. The case $N = 3$ is studied fully for $r = (1, 2, 3, 4)$. Basic aspects for all (N, r) are discussed. Full exploitation of such symmetries lead to a formalism quite different from, possibly generalized, algebraic Bethe ansatz. Chain Hamiltonians are studied. The specific types of spin flips they induce and propagate are pointed out. The inverse Cayley transform of the YB matrix giving the potential leading to factorizable S-matrix is constructed explicitly for $N = 3$ as also the full set of $\hat{R}tt$ relations. Perspectives are discussed in a final section.

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1 Introduction

New classes of braided matrices were presented in recent papers [1, 2]. Statistical models corresponding to [1] have been presented in [3]. Here we present those corresponding to [2]. Different types of statistical models thus obtained will be compared at the end (sec. 7). In [2] two distinct classes of braid matrices $(\hat{o}_N, \hat{p}_N)$ were presented. Here we consider only the $\hat{o}_N$ $(N \geq 3)$. For real, positive values of the parameter q and a certain domain (depending on q and N) of the spectral parameter θ, one obtains $N^2 \times N^2$ braid matrices with all nonzero elements real, positive giving nonnegative Boltzmann weights. For the class $\hat{p}_N$ one encounters both positive and negative elements and thus one would need suitable reinterpretation of the corresponding Boltzmann weights.

We first recapitulate briefly the $\hat{o}_N$ braid matrices [2]. The $N^2 \times N^2$ baxterized braid matrices satisfying (in standard notations)

$$\hat{R}_{12}(\theta)\, \hat{R}_{23}(\theta + \theta')\, \hat{R}_{12}(\theta') = \hat{R}_{23}(\theta')\, \hat{R}_{12}(\theta + \theta')\, \hat{R}_{23}(\theta) \tag{1.1}$$

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are given by

$$\hat{R}(\theta) = I - \frac{\sinh\theta}{\sinh(\eta+\theta)} P'_0, \tag{1.2}$$

where

$$e^{\eta} + e^{-\eta} = [N-1] + 1 \equiv \frac{q^{N-1} - q^{-N+1}}{q - q^{-1}} + 1 \tag{1.3}$$

and

$$P'_0 = \sum_{i,j=1}^{N} q^{\rho_{j'} - \rho_j} (ij) \otimes (i'j') \tag{1.4}$$

with the follwing notations:

1. The $N \times N$ matrix (ij) has only one non-zero element, unity, on row i and column j and

$$(i', j') = (N - i + 1, N - j + 1). \tag{1.5}$$

2. The N-tuple $(\rho_1, \rho_2, \ldots, \rho_N)$ is defined as

$$\left(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{1}{2}\right) \tag{1.6}$$

for $N = 2n + 1$ and

$$(n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, -n+1) \tag{1.7}$$

for $N = 2n$.

Of the three projectors (P_+, P_-, P_0) providing a spectral resolution of $SO_q(N)$ braid matrices only

$$P'_0 = ([N-1] + 1) P_0 \tag{1.8}$$

appears in our class. To signal this provenance (along with crucial differences) our class is designated as $\hat{o}_N$. More relevant discussions can be found in [2].

We now introduce the permutation matrix

$$P = \sum_{i,j} (ij) \otimes (ji), \qquad P^2 = I \tag{1.9}$$

the Yang-Baxter matrix

$$R(\theta) = P\hat{R}(\theta) \tag{1.10}$$

and the monodromy matrices satisfying

$$\hat{R}(\theta - \theta')(t(\theta) \otimes t(\theta')) = (t(\theta') \otimes t(\theta))\hat{R}(\theta - \theta') \tag{1.11}$$

The t-matrix satisfying (1.11) is $N \times N$ in terms of the blocks

$$t_{ij}, \qquad (i, j = 1, \ldots, N) \tag{1.12}$$

each t_{ij} being itself a matrix whose dimension is prescribed as follows. One starts with $N \times N$ blocks t_{ij} obtained from the standard prescription (satisfying (1.11))

$$t^{(1)}(\theta) = P\hat{R}(\theta) = R(\theta) \tag{1.13}$$

and then a hierarchy is obtained implementing the coproduct prescription

$$t_{ij}^{(r)}(\theta) = \sum_{k_1,\ldots,k_{r-1}} t_{ik_1}^{(1)}(\theta) \otimes t_{k_1k_2}^{(1)}(\theta) \otimes \cdots \otimes t_{k_{r-1},j}^{(1)}(\theta). \tag{1.14}$$

Starting with (1.13), this prescription assures that $t^{(r)}(\theta)$ satisfies (1.11).

Now the transfer matrix is defined, for each order r, as

$$T^{(r)}(\theta) = \sum_{i=1}^{N} t_{ii}^{(r)}(\theta). \tag{1.15}$$

The trace and more generally the eigenstates and the eigenvalues of $T^{(r)}(\theta)$ provide crucial properties of the statistical mechanical model associated with $\hat{R}(\theta)$. In particular, (1.1), (1.11), (1.13), (1.14), (1.15) all together assure the commutativity

$$[T(\theta), T(\theta')] = 0. \tag{1.16}$$

Commutative transfer matrices provide the crucial feature of exactly solvable models of statistical mechanics, the braid matrices encoding star-triangle relations [4]. For our specific case ($\hat{o}_N$) we illustrate, in the following section, some basic features for the simplest case ($N = 3$). Certain aspects for $N > 3$ will be presented afterwards (sec. 5).

Define

$$K(\theta) = -\frac{\sinh\theta}{\sinh(\eta+\theta)}, \tag{1.17}$$

where (setting $N = 3$ in (1.3))

$$e^{\eta} + e^{-\eta} = q + q^{-1} + 1. \tag{1.18}$$

For

$$-\eta < \theta < 0, \qquad K(\theta) > 0. \tag{1.19}$$

For

$$\begin{aligned} \theta &= 0, & K(0) &= 0, \\ \theta &= -\frac{\eta}{2}, & K\left(-\frac{\eta}{2}\right) &= 1, \\ \theta &\longrightarrow -\eta, & K(\theta) &\longrightarrow +\infty. \end{aligned} \tag{1.20}$$

Henceforward we consider the domain (1.19).

2 Trace of the transfer matrix from iterative structure

The standard prescription (1.13) yields for $\hat{o}_3$

$$t^{(1)}(\theta) = P\hat{R}(\theta) = P\left(I + K(\theta)P_0'\right) \tag{2.1}$$

and hence (suppressing now the argument θ for simplicity)

$$\begin{gathered} t_{11}^{(1)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K \end{vmatrix}, \qquad t_{12}^{(1)} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q^{1/2}K & 0 \end{vmatrix}, \qquad t_{13}^{(1)} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1+qK & 0 & 0 \end{vmatrix}, \\ t_{21}^{(1)} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & q^{-1/2}K \\ 0 & 0 & 0 \end{vmatrix}, \qquad t_{22}^{(1)} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1+K & 0 \\ 0 & 0 & 0 \end{vmatrix}, \qquad t_{23}^{(1)} = \begin{vmatrix} 0 & 0 & 0 \\ q^{1/2}K & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}, \\ t_{31}^{(1)} = \begin{vmatrix} 0 & 0 & 1+q^{-1}K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \qquad t_{32}^{(1)} = \begin{vmatrix} 0 & q^{-1/2}K & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}, \qquad t_{33}^{(1)} = \begin{vmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}. \end{gathered} \tag{2.2}$$

All θ-dependence is contained in the parameter K, as defined in (1.17). Starting with the 3×3 blocks the prescription (1.14) gives $3^r \times 3^r$ blocks $t_{ij}^{(r)}$. The recursion relations for our case ($N = 3$) are (for $j = 1, 2, 3$)

$$t_{1j}^{(r+1)} = \left| \begin{array}{ccc} t_{1j}^{(r)} & 0 & 0 \\ t_{2j}^{(r)} & 0 & 0 \\ (1+qK)\, t_{3j}^{(r)} & q^{1/2} K t_{2j}^{(r)} & K t_{1j}^{(r)} \end{array} \right|, \tag{2.3}$$

$$t_{2j}^{(r+1)} = \left| \begin{array}{ccc} 0 & t_{1j}^{(r)} & 0 \\ q^{1/2} K t_{3j}^{(r)} & (1+K)\, t_{2j}^{(r)} & q^{-1/2} K t_{1j}^{(r)} \\ 0 & t_{3j}^{(r)} & 0 \end{array} \right|, \tag{2.4}$$

$$t_{3j}^{(r+1)} = \left| \begin{array}{ccc} K t_{3j}^{(r)} & q^{-1/2} K t_{2j}^{(r)} & (1+q^{-1}K)\, t_{1j}^{(r)} \\ 0 & 0 & t_{2j}^{(r)} \\ 0 & 0 & t_{3j}^{(r)} \end{array} \right|. \tag{2.5}$$

The transfer matrix is iterated as

$$\begin{aligned} T^{(r+1)} &= t_{11}^{(r+1)} + t_{22}^{(r+1)} + t_{33}^{(r+1)} \\ &= \left| \begin{array}{ccc} t_{11}^{(r)} + K t_{33}^{(r)} & t_{12}^{(r)} + q^{-1/2} K t_{23}^{(r)} & (1+q^{-1}K)\, t_{13}^{(r)} \\ t_{21}^{(r)} + q^{1/2} K t_{32}^{(r)} & (1+K)\, t_{22}^{(r)} & q^{-1/2} K t_{12}^{(r)} + t_{23}^{(r)} \\ (1+qK)\, t_{31}^{(r)} & q^{1/2} K t_{21}^{(r)} + t_{32}^{(r)} & K t_{11}^{(r)} + t_{33}^{(r)} \end{array} \right|. \end{aligned} \tag{2.6}$$

Hence

$$\begin{aligned} \mathrm{Tr}\left(T^{(r+1)}\right) &= \mathrm{Tr}\left(t_{11}^{(r)} + K t_{33}^{(r)} + (1+K)\, t_{22}^{(r)} + K t_{11}^{(r)} + t_{33}^{(r)}\right) \\ &= (1+K)\, \mathrm{Tr}\left(t_{11}^{(r)} + t_{22}^{(r)} + t_{33}^{(r)}\right) \\ \mathrm{Tr}\left(T^{(r+1)}\right) &= (1+K)\, \mathrm{Tr}\left(T^{(r)}\right). \end{aligned} \tag{2.7}$$

But from (2.2),

$$\mathrm{Tr}\left(T^{(1)}\right) = \mathrm{Tr}\left(t_{11}^{(1)} + t_{22}^{(1)} + t_{33}^{(1)}\right) = 3\,(1+K). \tag{2.8}$$

Hence

$$\mathrm{Tr}\left(T^{(r)}\right) = 3\,(1+K)^r. \tag{2.9}$$

Thus we obtain the trace of $T^{(r)}$ for all r directly without constructing explicitly the eigenstates and the 3^r eigenvalues. But the latter being of crucial interest we now turn to their systematic explicit constructions.

3 Eigenstates and eigenvalues $(N = 3)$

For $N = 3$ the transfer matrix $T^{(r)}(\theta)$ of order r acts on a space of dimension 3^r. Construction of eigenstates corresponds to diagonalization of $T^{(r)}$ on such a base space. But basic symmetries of $T^{(r)}$ (Sec. 1) for our case have profound consequences. They reduce the problem so that on has effectively to diagonalize subspaces whose dimensions increase polynomially with r (rather than according to the power law 3^r). To formulate these features conveniently we introduce the following conventions for state-labels.

For the fundamental case, $r = 1$, the 3-dimensional basis is denoted as

$$|1\rangle \equiv \left| \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right\rangle, \quad |2\rangle \equiv \left| \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right\rangle, \quad |3\rangle \equiv \left| \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right\rangle. \tag{3.1}$$

For $r>1$, the order of the indices (1,2,3) represents the tensored structure. Thus, for example, for $r=5$,

$$|11231\rangle \equiv |1\rangle \otimes |1\rangle \otimes |2\rangle \otimes |3\rangle \otimes |1\rangle . \tag{3.2}$$

The fundamental realizations $t_{ij}^{(1)}$ of (2.2) implemented in the tensored structure (1.14) of $t_{ij}^{(r)}$ lead to the following major consequences.

(I): Each set of states corresponding to given sum of the indices (state labels 1,2,3) forms a closed subspace under the action of $T^{(r)}(\theta)$. Define with (with $a_i = (1,2,3)$)

$$W|a_1 a_2 \ldots a_r\rangle = (a_1 + a_2 + \cdots + a_r)|a_1 a_2 \ldots a_r\rangle . \tag{3.3}$$

Then

$$\left[T^{(r)}(\theta), W\right] = 0 \tag{3.4}$$

implying for each state on the right of

$$T^{(r)}(\theta)|a_1 a_2 \ldots a_r\rangle = \sum_{b_i} f_{(a,b)}(\theta)|b_1 b_2 \ldots b_r\rangle , \qquad (b_1 + b_2 + \cdots + b_r) = (a_1 + a_2 + \cdots + a_r) . \tag{3.5}$$

Thus the 3^r dimensional base space of $T^{(r)}$ splits into $(2r+1)$ closed subspaces under the action of $T^{(r)}$ as

$$S_r,\ S_{r+1}, \ldots,\ S_{2r-1},\ S_{2r},\ S_{2r+1}, \ldots,\ S_{3r}, \tag{3.6}$$

where S_n corresponds to $a_1 + a_2 + \cdots + a_r = n$. In constructing eigenstates of $T^{(r)}$ each S_n can be treated separately simplifying the problem considerably. The simplest subspaces are the extreme ones, namely

$$S_r = |11\ldots 1\rangle \tag{3.7}$$

and

$$S_{3r} = |33\ldots 3\rangle \tag{3.8}$$

(the index 1(3) being repeated r times). These are already automatically eigenstates. The highest dimensional subspace is obtained for $n=2r$ which includes the state $|22\ldots 2\rangle$. Special feature of some subspaces will be displayed below.

(II): Within each subspace again $T^{(r)}(\theta)$ commutes with circular permutations of states labels. Thus (CP) representing a circular permutation,

$$\left[T^{(r)}(\theta), (CP)\right] = 0 \tag{3.9}$$

in the sense

$$(CP)^2 T^{(r)}|a_1 a_2 a_3 \ldots a_{r-1} a_r\rangle = (CP)T^{(r)}|a_r a_1 a_2 \ldots a_{r-2} a_{r-1}\rangle = T^{(r)}|a_{r-1} a_r a_1 \ldots a_{r-3} a_{r-2}\rangle \tag{3.10}$$

and so on for all successive (CP) of the indices $(a_1 a_2 \ldots a_r)$.

(III): As a consequence the states in each invariant subspace can again be grouped together implementing roots of unity as follows. Let ω be any r-th root of unity, i.e.

$$\omega = \left(1, e^{i\frac{2\pi}{r}}, e^{i\frac{2\pi}{r}\cdot 2}, \ldots, e^{i\frac{2\pi}{r}\cdot(r-1)}\right) \tag{3.11}$$

and (for each possible value of ω, separately)

$$\begin{aligned} |a_1 a_2 a_3 \ldots a_{r-1} a_r\rangle_\omega \equiv{}& |a_1 a_2 a_3 \ldots a_{r-1} a_r\rangle + \omega|a_r a_1 a_2 \ldots a_{r-2} a_{r-1}\rangle + \omega^2|a_{r-1} a_r a_1 \ldots a_{r-3} a_{r-2}\rangle \\ &+ \cdots + \omega^{r-1}|a_2 a_3 a_4 \ldots a_r a_1\rangle \end{aligned} \tag{3.12}$$

The components states are, evidently, all in the same invariant subspace. For r different values of ω these provide a mutually orthogonal set of r states diagonalizing (CP) since

$$(CP)\,|a_1a_2a_3\dots a_{r-1}a_r\rangle_\omega = \omega\,|a_1a_2a_3\dots a_{r-1}a_r\rangle_\omega \tag{3.13}$$

The action of $T^{(r)}$ on, say, $|a_1a_2\dots a_{r-1}a_r\rangle$ gives directly, due to (3.10), that on $|a_1a_2\dots a_{r-1}a_r\rangle_\omega$ for all values of ω. Thus one can effectively reduce the dimension of the relevant subspace S_n for a given sum of state labels, $(a_1+a_2+\cdots+a_{r-1}+a_r)=n$. Such a "two-step reduction", firstly restriction to invariant subspaces S_n, secondly introduction of roots of unity to form eigenstates of (CP) will be shown to lead to a much slower increase with r (as compared to $e^{(\ln 3)r}$) of the dimension of the spaces on which one has to diagonalize $T^{(r)}$. This will be first displayed through particular examples. The general formulation will be given at the end of this section.

(IV): But another symmetry is appropriately mentioned at this stage (to be illustrated later explicitly). Interchanging the indices as

$$(1,2,3)\longrightarrow(3,2,1) \tag{3.14}$$

The action of $T^{(r)}$ is directly obtained via the inversion

$$q\longrightarrow q^{-1} \tag{3.15}$$

in each coefficient. Thus the invariant subspaces related through (3.14) need not ne studied separately. The corresponding eigenstates and eigenvalues are related through (3.15). It is sufficient to study the first $(r+1)$ subspaces since under (3.14) and (3.15),

$$S_{2r}\longrightarrow S_{2r},\qquad (S_r,S_{r+1},\dots,S_{2r-1})\rightleftharpoons(S_{3r},S_{3r-1},\dots,S_{2r+1})\,. \tag{3.16}$$

Explicit examples for $r=(3,4)$ will follow. Our $\hat{o}_N$ braid matrices remain nontrivial for $q=1$ as pointed out in Ref. 2. Now (3.14) becomes a full symmetry. The degeneracy thus induced is of interest.

(V): A final crucial feature is due to (1.16),

$$\left[T^{(r)}(\theta)\,,T^{(r)}(\theta')\right]=0. \tag{3.17}$$

Suppose that for, say, $r=4$ in some subspace one obtains a closed subset of states (A,B,C,D) with

$$T^{(4)}(\theta)\,A=a_{11}A+a_{12}B+a_{13}C+a_{14}D,\dots,\;T^{(4)}(\theta)\,D=d_{11}A+d_{12}B+d_{13}C+d_{14}D. \tag{3.18}$$

The coefficients $(a_{11},\dots,d_{14})$ are in general polynomials in $K(\theta)$, the maximal degree being $r=4$ for this case. Define eigenstates as

$$T^{(4)}(\theta)\,(\alpha A+\beta B+\gamma C+\delta D)=v\,(\alpha A+\beta B+\gamma C+\delta D) \tag{3.19}$$

which are to be solved for by implementing (3.18) on the left. Consistency with (3.17) imposes θ-independence (K-independence) of $(\alpha,\beta,\gamma,\delta)$. Hence on the right only v can be K-dependent. All K-dependence of $(a_{11},\dots,d_{14})$ on the left must thus factorize as a polynomial (here for $r=4$)

$$v=f_4K^4+f_3K^3+f_2K^2+f_1K+f_0 \tag{3.20}$$

for suitable (f_4,f_3,f_2,f_1,f_0) which can depend on (q,ω) only. In general this leads to a set of overdetermined set of coupled linear equations (for our case) in

$$(\alpha,\beta,\gamma,\delta;f_4,f_3,f_2,f_1,f_0) \tag{3.21}$$

Varied illustrations will follows. Moreover, while all eigenvalues are, in general, K- and q-dependent, all explicit q-dependence (except for the implicit one through K of (1.17) and (1.18)) must cancel in the overall trace (summing over all subspaces) to give (2.9), i.e.

$$\mathrm{Tr}\left(T^{(r)}\right) = 3\,(1+K)^r\,. \tag{3.22}$$

This provides stringent check (Appendix A).

Special features of the subspaces (S_r, S_{3r}), (S_{r+1}, S_{3r-1}), S_{2r}:

• $\underline{(S_r, S_{3r})}$**:** As mentioned following (3.7), (3.8) these two are 1-dimensional subspaces. One obtains immediately, for all r,

$$T^{(r)}\,(\theta)\,|11\ldots1\rangle = (1+K^r)\,|11\ldots1\rangle \tag{3.23}$$
$$T^{(r)}\,(\theta)\,|33\ldots3\rangle = (1+K^r)\,|33\ldots3\rangle \tag{3.24}$$

These eigenstates of (CP), singlets, provide the simplest illustrations of (3.14), (3.15).

• $\underline{(S_{r+1}, S_{3r-1})}$**:** For arbitrary r, with

$$\omega = \left(1, e^{i\frac{2\pi}{r}}, e^{i\frac{2\pi}{r}\cdot 2}, \ldots, e^{i\frac{2\pi}{r}\cdot(r-1)}\right) \tag{3.25}$$

define

$$X_\omega = |111\ldots12\rangle + \omega\,|211\ldots11\rangle + \omega^2\,|121\ldots11\rangle + \omega^{r-1}\,|111\ldots21\rangle\,, \tag{3.26}$$
$$Y_\omega = |333\ldots32\rangle + \omega\,|233\ldots33\rangle + \omega^2\,|323\ldots33\rangle + \omega^{r-1}\,|333\ldots23\rangle\,. \tag{3.27}$$

One easily obtains

$$T^{(r)}\,(\theta)\,X_\omega = \left(K^r\omega + \omega^{r-1}\right)X_\omega, \tag{3.28}$$
$$T^{(r)}\,(\theta)\,Y_\omega = \left(K^r\omega + \omega^{r-1}\right)Y_\omega. \tag{3.29}$$

For the r values of ω one obtains thus, in a single stroke, all the requisite r eigenstates for these two r-dimensional subspaces. Note that

$$\sum_\omega \left(K^r\omega + \omega^{r-1}\right) = \sum_\omega \left(K^r\omega + \omega^{-1}\right) = 0. \tag{3.30}$$

Hence (S_{r+1}, S_{3r-1}) do not contribute to the total trace $Tr\left(T^{(r)}\,(\theta)\right)$.

For S_{r+2} (S_{3r-2}) already the structure of eigenstates and eigenvalues are not so simple. (See App. A for $r = 3, 4$). Some special features of S_{2r} are however worth mentioning, particularly to compare the structures of r prime and non-prime.

• $\underline{(S_{2r})}$**:** Like $|11\ldots1\rangle$ and $|33\ldots3\rangle$, $|22\ldots2\rangle$ is also a singlet under (CP). But unlike the former the latter one does not form an 1-dimensional subspace. It can get coupled with the other states of S_{2r} (for $\omega = 1$) as follows. When r is prime, apart form $|22\ldots2\rangle$, S_{2r} is composed of r-plets (formed using ω with $\omega^r = 1$). When r is factorizable there can be intermediate multiplets corresponding to factors $(n_1, n_2, \ldots, n_k)$ of $r = (n_1 n_2 \cdots n_k)$. Thus for $r = 4$ (the first factorizable r) there are doublets corresponding to $r = 2\cdot 2$. For $r = 6$, there are doublets and triplets between $1-$ and $6-$plets. Let us illustrate the situation using the simplest non-trivial cases $r = 3, 4$.

$\diamond$ $\underline{(r=3, S_6)}$**:** Define

$$A_1 = |222\rangle\,, \qquad B_\omega = |123\rangle + \omega\,|312\rangle + \omega^2\,|231\rangle\,, \qquad C_\omega = |321\rangle + \omega\,|132\rangle + \omega^2\,|213\rangle\,, \tag{3.31}$$

where $\omega = \left(1, e^{i\frac{2\pi}{3}}, e^{i\frac{2\pi}{3}\cdot 2}\right)$. In our notation A_1 indicates that here (for singlet) one has only $\omega = 1$. Correspondingly (B_1, C_1) will denote the latter for $\omega = 1$. Consistently with (3.9) set

$$T^{(3)}(\theta)\left(\alpha A_1 + \beta B_1 + \gamma C_1\right) = v\left(\alpha A_1 + \beta B_1 + \gamma C_1\right) \tag{3.32}$$

for $\omega = 1$ and

$$T^{(3)}(\theta)\left(\mu B_\omega + \nu C_\omega\right) = w\left(\mu B_\omega + \nu C_\omega\right) \tag{3.33}$$

for $\omega = \left(e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}\right)$. Here (v, w) are assumed to be cubic polynomials in K and (α, β, γ), (μ, ν) to be K-independent. Note also that

$$(1 \rightleftharpoons 3)\, B_\omega = C_\omega. \tag{3.34}$$

Hence (consistently with (3.14), (3.15)) one obtains the coefficients in $T^{(3)}(\theta)\, C_\omega$ by inverting q to q^{-1} in those of $T^{(3)}(\theta)\, B_\omega$. Explicit solutions are given in App. A. Here we only note that the decoupling of A_1 in (3.33) is assured via the structure

$$\begin{aligned}
T^{(3)}(\theta)\, A_1 &= a_{11} A_1 + a_{12} B_1 + a_{13} C_1 \\
T^{(3)}(\theta)\, B_\omega &= \left(1 + \omega + \omega^2\right) b_{11} A_1 + b_{12} B_\omega + b_{13} C_\omega \\
T^{(3)}(\theta)\, C_\omega &= \left(1 + \omega + \omega^2\right) c_{11} A_1 + c_{12} B_\omega + c_{13} C_\omega,
\end{aligned} \tag{3.35}$$

$\diamond$ $\underline{(r=4, S_8)}$**:** Here, after the (CP)-singlet

$$A_1 = |2222\rangle \tag{3.36}$$

one has also the doublets

$$B_{\pm 1} = |1313\rangle \pm |3131\rangle \tag{3.37}$$

and then the quartets completing the 19 dimensional S_8 for all values of ω, namely,

$$\omega = \left(1, e^{i\frac{2\pi}{4}}, e^{i\frac{2\pi}{4}\cdot 2}, e^{i\frac{2\pi}{4}\cdot 3}\right) = (1, i, -1, -i) \tag{3.38}$$

$$\begin{aligned}
C_\omega &= |1133\rangle + \omega\,|3113\rangle + \omega^2\,|3311\rangle + \omega^3\,|1331\rangle\,, \\
D_\omega &= |1223\rangle + \omega\,|3122\rangle + \omega^2\,|2312\rangle + \omega^3\,|2231\rangle\,, \\
E_\omega &= |3221\rangle + \omega\,|1322\rangle + \omega^2\,|2132\rangle + \omega^3\,|2213\rangle\,, \\
F_\omega &= |1232\rangle + \omega\,|2123\rangle + \omega^2\,|3212\rangle + \omega^3\,|2321\rangle\,.
\end{aligned} \tag{3.39}$$

Note also that

$$(1 \rightleftharpoons 3)\left(C_\omega, D_\omega, E_\omega, F_\omega\right) = \left(\omega^2 C_\omega, E_\omega, D_\omega, \omega^2 F_\omega\right) \tag{3.40}$$

which simplifies computations according to (3.14), (3.15). The set F_ω alone has a distinctive feature. The two indices 2 remain separated (unlike for D_ω, E_ω) under (CP). This singles it out as directly an eigenstates of $T^{(4)}(\theta)$ (App. A). As for $(C_\omega, D_\omega, E_\omega)$ decouplings, analogous to (3.35) but in two stages

(1) from A_1 for $\omega = (-1, \pm i)$

(2) and also from $B_{\pm 1}$ for $\omega = (\pm i)$

are assured through factors of the type (App. A)

$$\left(1+\omega+\omega^2+\omega^3\right),\ \left(1+\omega^2\right)(1\pm\omega)\,. \tag{3.41}$$

The maximal set of 5 coupled linear equations arises for ($\omega=1$)

$$T^{(4)}(\theta)\left(aA_1+bB_1+cC_1+dD_1+eE_1\right)=v_1\left(aA_1+bB_1+cC_1+dD_1+eE_1\right). \tag{3.42}$$

To conclude we emphasize again that for $r=(3,4)$ in base spaces respectively of dimensions (27,81) the maximal set of coupled linear equations encountered are sets of (3,5) respectively. This is the slow growth with r signalled before (end of (III)).

For $r=(1,2,3,4)$ we have studied the invariant subspaces S_n explicitly. Let us now indicate the general situation. Associate the variables (x_1,x_2,x_3) to the states $(|1\rangle,|2\rangle,|3\rangle)$ respectively. In the expansion

$$(x_1+x_2+x_3)^r=\sum_{n_1,n_2,n_3}C_{n_1,n_2,n_3}x_1^{n_1}x_2^{n_2}x_3^{n_3}. \tag{3.43}$$

for each term

$$n_1+n_2+n_3=r \tag{3.44}$$

and

$$\sum_{n_1,n_2,n_3}C_{n_1,n_2,n_3}=3^r. \tag{3.45}$$

Imposing an additional constraint one obtains the subsets

$$\dim S_n=\sum_{n_1,n_2,n_3}C_{n_1,n_2,n_3},\qquad (n_1+2n_2+3n_3=n) \tag{3.46}$$

for $n=(r,r+1,\ldots,2r,\ldots,3r)$. The dimension of the total base space for order r is given by (3.45).

Let us consider as an example the central subspace S_{10} for $r=5$. From (3.46) one easily finds

$$\dim S_{10}=51\qquad (r=5)\,. \tag{3.47}$$

The states can be grouped into multiplets as follows with

$$\omega^5=1, \tag{3.48}$$
$$\begin{aligned}
V_1^{(1)}&=|22222\rangle\,,\\
V_2^{(\omega)}&=\left(|13222\rangle+\omega\,|21322\rangle+\omega^2\,|22132\rangle+\omega^3\,|22213\rangle+\omega^4\,|32221\rangle\right)\equiv((CP)\,|13222\rangle)_\omega\,,\\
V_3^{(\omega)}&=((CP)\,|12322\rangle)_\omega\,,\\
V_4^{(\omega)}&=((CP)\,|13132\rangle)_\omega\,,\\
V_5^{(\omega)}&=((CP)\,|13312\rangle)_\omega\,,\\
V_6^{(\omega)}&=((CP)\,|13321\rangle)_\omega\,,\\
\left(V_7^{(\omega)},\ldots,V_{11}^{(\omega)}\right)&=(1\rightleftharpoons 3)\left(V_2^{(\omega)},\ldots,V_6^{(\omega)}\right),
\end{aligned} \tag{3.49}$$

i.e. $V_7^{(\omega)}=((CP)\,|31222\rangle)_\omega$ and so on. For $\omega=1$, now one has to solve a set of 11 (coupling $V_1^{(1)},\ldots V_{11}^{(1)}$) linear equations. This is the maximal such set for $r=5$ where the total dimension is 243.

Whenever r is a prime number, i.e. $r=(1,3,5,7,11,13,\ldots)$, the multiplet structure is relatively simple. Thus for S_{2r} apart from $|22\ldots2\rangle$ there are only r-plets in terms of the roots $\omega^r=1$. When r is factorizable lower multiplets can arise corresponding to factors of r. We have illustrated this for $r=4$.

4 Chain Hamiltonians ($N = 3$)

The Hamiltonian for order r is defined as

$$H^{(r)} = \left(T^{(r)}(\theta)\right)^{-1}_{\theta=0}\left(\partial_\theta\left(T^{(r)}(\theta)\right)\right)_{\theta=0} \tag{4.1}$$

Instead of using the standard formulation as a sum (see the basic references in sec. 4 of Ref. 3)

$$H^{(r)} = \sum_{k=1}^{r} I \otimes I \otimes \cdots \otimes \dot{\hat{R}}_{k,k+1}(0) \otimes I \otimes \cdots \otimes I, \tag{4.2}$$

where

$$\dot{\hat{R}}_{k,k+1}(0) = \left(\partial_\theta \hat{R}_{k,k+1}(\theta)\right)_{\theta=0} \tag{4.3}$$

with the circular boundary condition for $k = r$ ($r + 1 \approx 1$) we we will use (4.1) directly, as explained below, in a fashion particularly well-adapted to our formalism for constructing eigenstates.

Define starting from (1.17) i.e.

$$K(\theta) = -\frac{\sinh\theta}{\sinh(\eta+\theta)}, \tag{4.4}$$

$$\dot{K}_0 \equiv (\partial_\theta K(\theta))_{\theta=0} = -(\sinh\eta)^{-1} \tag{4.5}$$

with $K_0 = (K(\theta))_{\theta=0} = 0$. We start with eigenstate of $T^{(r)}(\theta)$

$$|V\rangle_\omega = (c_1 A_1 + c_2 A_2 + \cdots + c_m A_m)_\omega, \tag{4.6}$$

where the subscript ω indicates that each A_i ($i = 1, \ldots, m$) is an eigenstate of (CP), circular permutation of r state labels corresponding to a subspace S_n ($n = r, \ldots, 3r$). (See Sec. 3 and App. A). Thus for example, for $r = 3$ and $S_n = S_5$ (see (A.13) and (A.18)-(A.20)) $|V\rangle_\omega = aA_\omega + bB_\omega$, where

$$A_\omega = \left(|113\rangle + \omega|311\rangle + \omega^2|131\rangle\right), \qquad B_\omega = \left(|122\rangle + \omega|212\rangle + \omega^2|221\rangle\right), \tag{4.7}$$

with $\omega^3 = 1$. Quite generally, if for (4.6)

$$T^{(r)}(\theta)|V\rangle = v|V\rangle = v\left(\sum_k c_k A_k\right) \tag{4.8}$$

then as explained and emphasized (in sec. 3 and App. A) the coefficients c_k can depend on q (but not on θ) the only θ-dependence on the right is in v, a polynomial of order r in $K(\theta)$,

$$v = f_r(K(\theta))^r + f_{r-1}(K(\theta))^{r-1} + \cdots + f_1(K(\theta)) + f_0, \tag{4.9}$$

where the coefficients f_i are each θ-independent. Thus for (4.7) the solutions (for each value of ω) are

(1) $\quad (a,b) = \left(q^{1/2} + \omega q^{-1/2}, 1\right),$ (4.10)

$$v = \omega^2 K^3 + \left(\left(q+q^{-1}\right)\omega^2 + \left(1+\omega+\omega^2\right)\right)K^2 + \left(\left(q+q^{-1}\right)\omega + \left(1+\omega+\omega^2\right)\right)K + \omega$$

(2) $\quad (a,b) = \left(1, -\left(q^{1/2} + \omega^2 q^{-1/2}\right)\right), \qquad v = \omega^2 K^3 + \omega.$ (4.11)

From (4.5), (4.8) and (4.9) one obtains (since $K_0 = 0$) the general result (with $\dot{T}^{(r)}_0 \equiv \left(\partial_\theta T^{(r)}(\theta)\right)_{\theta=0}$, $T^{(r)}_0 = \left(T^{(r)}(\theta)\right)_{\theta=0}$)

$$\dot{T}^{(r)}_0|V\rangle = \dot{K}_0 f_1|V\rangle, \tag{4.12}$$

$$T^{(r)}_0|V\rangle = f_0|V\rangle = \omega|V\rangle \tag{4.13}$$

and hence

$$\left(T_0^{(r)}\right)^{-1}|V\rangle = \omega^{r-1}|V\rangle \tag{4.14}$$

The result $f_0 = \omega$ (and $f_0^{-1} = \omega^{r-1}$ for $\omega^r = 1$) is a general one. This corresponds to our use eigenstates of (CP) as basis states since for our class T_0 coincides with (CP). Hence finally

$$H^{(r)}|V\rangle = T_0^{-1}\dot{T}_0|V\rangle = \left(\dot{K}_0\omega^{r-1}f_1\right)|V\rangle \tag{4.15}$$

Thus starting with an eigenstate of $T^{(r)}(\theta)$ in our formalism it remains one of $H^{(r)}$ and the eigenvalue of $H^{(r)}$ is extracted, as above from that of $T^{(r)}$. Note that for

$$f_1 = 0, \qquad H^{(r)}|V\rangle = 0. \tag{4.16}$$

Thus for (4.11)

$$H^{(3)}\left(A_\omega - \left(q^{1/2} + \omega^2 q^{-1/2}\right)B_\omega\right) = 0. \tag{4.17}$$

From (3.23)-(3.29) it follows that, for all r,

$$H^{(r)}\left(S_r, S_{3r}; S_{r+1}, S_{3r-1}\right) \approx 0, \tag{4.18}$$

i.e. each eigenstate belonging to these subspaces is annihilated by $H^{(r)}$.

For $r = 2$, the explicit form of the Hamiltonian is

$$\begin{aligned}\left(\dot{K}_0\right)^{-1} H^{(2)} = {} & \left(q + q^{-1}\right)(11)\otimes(33) + \left(q^{1/2} + q^{-1/2}\right)(12)\otimes(32) + 2\,(13)\otimes(31) \\ & + \left(q^{1/2} + q^{-1/2}\right)(21)\otimes(23) + 2\,(22)\otimes(22) + \left(q^{1/2} + q^{-1/2}\right)(23)\otimes(21) \\ & + 2\,(31)\otimes(33) + \left(q^{1/2} + q^{-1/2}\right)(32)\otimes(12) + \left(q + q^{-1}\right)(33)\otimes(11)\end{aligned} \tag{4.19}$$

Consistently with (4.18)

$$H^{(2)}\left(|11\rangle, |33\rangle; |12\rangle, |21\rangle; |23\rangle, |32\rangle\right) = 0. \tag{4.20}$$

For the only remaining subspace S_4, setting

$$H^{(2)}\left(a|13\rangle + b|22\rangle + c|31\rangle\right) = v_H\left(a|13\rangle + b|22\rangle + c|31\rangle\right). \tag{4.21}$$

One obtains the solutions

$$\begin{aligned} &\textbf{(1)} \qquad (a,b,c) = \left(1, -\left(q^{1/2} + q^{-1/2}\right), 1\right);\ v_H = 0, \\ &\textbf{(2)} \qquad (a,b,c) = (1, 0, -1);\ v_H = \dot{K}_0\left(q + q^{-1} - 2\right), \\ &\textbf{(3)} \qquad (a,b,c) = \left(1, 2\left(q^{1/2} + q^{-1/2}\right)^{-1}, 1\right);\ v_H = \dot{K}_0\left(q + q^{-1} + 4\right). \end{aligned} \tag{4.22}$$

Combining (4.15) with (A.4)-(A.6) one consistently reproduces the results (4.20)-(4.22) obtained using the explicit form (4.19). Note that for $r = 2$ and $\omega = -1$, (4.15) gives

$$H^{(2)}|V\rangle = -\dot{K}_0 f_1|V\rangle. \tag{4.23}$$

This corresponds to the positive sign in solution (2) of (4.22) since in (A.6) the corresponding factor is

$$f_1 = -\left(q + q^{-1} - 2\right) \tag{4.24}$$

Such changes of sign introduce a qualitative change: $\mathrm{Tr}\left(T^{(r)}\right)$ in (2.9) has no explicit dependence on q (only an implicit one though K). But $\mathrm{Tr}\left(H^{(r)}\right)$ can have explicit q-dependence. For the simple example above $(N = 3, r = 2)$

$$\mathrm{Tr}\left(H^{(2)}\right) = 2\dot{K}_0\left(q + q^{-1} + 1\right) \tag{4.25}$$

Selection rules for transitions: Adopting the convention of attaching to the states $(|1\rangle, |2\rangle, |3\rangle)$ respectively the "spins"

$$(+, 0, -) \tag{4.26}$$

it is seen from (4.19) that the action of the Hamiltonian on neighboring sites, induces transitions only when the sum of the two spins is zero, i.e. for

$$(+-), \qquad (00), \qquad (-+) \tag{4.27}$$

The final states corresponding again to zero sum. Thus one has non-zero matrix elements for a neighboring pair $|ij\rangle \overset{H}{\rightarrow} |kl\rangle$ only when for the corresponding spins

$$\sigma_i + \sigma_j = 0 = \sigma_k + \sigma_l \tag{4.28}$$

Such matrix elements depend on $\left(\dot{K}_0, q\right)$. The structure of $H^{(r)}$ in (4.2) indicates that (4.21) is a generic feature. Any pair of the type (4.27) somewhere in the chain can start transitions which can propagate along the chain since the three possibilities in (4.27) can create such a pair with the next neighboring site and so on.

5 $N > 3$

Three basic features displayed and studied at length for $N = 3$ are:

(1) A simple recursion relation yielding the trace of the transfer matrix for any order r. (See (2.2)-(2.9)).

(2) Invariant subspaces corresponding to the sum of the state labels. (See (3.1)-(3.6) and App. A.)

(3) Role of (CP) circular permutation of state labels within each invariant subspace S_n. (See (3.9)-(3.13) and App. A.)

It was shown (for $N = 3$) how (2) and (3) greatly simplify the construction of eigenstates and eigenvalues of $T^{(r)}(\theta)$ for successive values of r.

We now indicate how these features are carried over for $N > 3$ via the simplest possibilities, namely $N = 4$, $r = (1, 2)$. Now, as compared to (1.17)-(1.18)

$$K(\theta) = -\frac{\sinh\theta}{\sinh(\eta + \theta)} \tag{5.1}$$

where $e^{\eta} + e^{-\eta} = (q^2 + 1 + q^{-2}) + 1 = (q + q^{-1})^2$. As compared to (2.2) (writing t_{ij} for $t_{ij}^{(1)}(\theta)$, $i = (1, 2, 3, 4)$ and K for $K(\theta)$) $t_{11} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K \end{vmatrix} \equiv (11) + K(44)$ and similarly,

$$\begin{aligned}
&t_{12} = (21) + Kq(43), \;\; t_{13} = (31) + Kq(42), \;\; t_{14} = \left(1 + Kq^2\right)(41), \\
&t_{21} = (12) + Kq^{-1}(34), \;\; t_{22} = (22) + K(33), \;\; t_{23} = (1 + K)(32), \;\; t_{24} = Kq(31) + (42), \\
&t_{31} = (13) + Kq^{-1}(24), \;\; t_{32} = (1 + K)(23), \;\; t_{33} = K(22) + (33), \;\; t_{34} = Kq(21) + (43), \\
&t_{41} = \left(1 + Kq^{-2}\right)(14), \;\; t_{42} = Kq^{-1}(13) + (24), \;\; t_{43} = Kq^{-1}(12) + (34), \;\; t_{44} = K(11) + (44)
\end{aligned} \tag{5.2}$$

As compared to (2.3)-(2.9) recursion relations are now (suppressing arguments θ)

$$t_{ij}^{(r+1)} = t_{i1}^{(1)} \otimes t_{1j}^{(r)} + t_{i2}^{(2)} \otimes t_{2j}^{(r)} + t_{i3}^{(1)} \otimes t_{3j}^{(r)} + t_{i4}^{(1)} \otimes t_{4j}^{(r)}, \qquad (i, j = 1, 2, 3, 4) \tag{5.3}$$

giving (due to (5.2))

$$\begin{aligned}
t_{1j}^{(r+1)} &= ((11)+K\,(44))\otimes t_{1j}^{(r)} + ((21)+Kq\,(43))\otimes t_{2j}^{(r)} + ((31)+Kq\,(42))\otimes t_{3j}^{(r)} \\
&\quad + \left(1+Kq^2\right)(41)\otimes t_{4j}^{(r)}, \\
t_{2j}^{(r+1)} &= \left((12)+Kq^{-1}\,(34)\right)\otimes t_{1j}^{(r)} + ((22)+K\,(33))\otimes t_{2j}^{(r)} + (1+K)\,(32)\otimes t_{3j}^{(r)} \\
&\quad + (Kq\,(31)+(42))\otimes t_{4j}^{(r)}, \\
t_{3j}^{(r+1)} &= \left((13)+Kq^{-1}\,(24)\right)\otimes t_{1j}^{(r)} + (1+K)\,(23)\otimes t_{2j}^{(r)} + (K\,(22)+(33))\otimes t_{3j}^{(r)} \\
&\quad + (Kq\,(21)+K\,(43))\otimes t_{4j}^{(r)}, \\
t_{4j}^{(r+1)} &= \left(1+Kq^{-2}\right)(14)\otimes t_{1j}^{(r)} + \left(Kq^{-1}\,(13)+(24)\right)\otimes t_{2j}^{(r)} + \left(Kq^{-1}\,(12)+(34)\right)\otimes t_{3j}^{(r)} \\
&\quad + (K\,(11)+(44))\otimes t_{4j}^{(r)}.
\end{aligned} \tag{5.4}$$

Hence for the transfer matrix

$$\begin{aligned}
T^{(r+1)} &= t_{11}^{(r+1)} + t_{22}^{(r+1)} + t_{33}^{(r+1)} + t_{44}^{(r+1)} \\
&= \left|\begin{array}{cccc}
t_{11}^{(r)}+Kt_{44}^{(r)} & t_{12}^{(r)}+Kq^{-1}t_{34}^{(r)} & t_{13}^{(r)}+Kq^{-1}t_{34}^{(r)} & \left(1+Kq^{-2}\right)t_{14}^{(r)} \\
t_{21}^{(r)}+Kt_{43}^{(r)} & t_{22}^{(r)}+Kt_{33}^{(r)} & (1+K)\,t_{23}^{(r)} & Kq^{-1}t_{13}^{(r)}+t_{24}^{(r)} \\
t_{31}^{(r)}+Kqt_{42}^{(r)} & (1+K)\,t_{32}^{(r)} & Kt_{22}^{(r)}+t_{33}^{(r)} & Kq^{-1}t_{12}^{(r)}+t_{34}^{(r)} \\
\left(1+Kq^{2}\right)t_{41}^{(r)} & t_{42}^{(r)}+Kqt_{31}^{(r)} & Kqt_{21}^{(r)}+t_{43}^{(r)} & Kt_{11}^{(r)}+t_{44}^{(r)}
\end{array}\right|.
\end{aligned} \tag{5.5}$$

One now obtains

$$\mathrm{Tr}\left(T^{(r+1)}\right) = (K+1)\,\mathrm{Tr}\left(T^{r}\right). \tag{5.6}$$

But

$$T^{(1)} = (K+1)\,I_4. \tag{5.7}$$

Hence

$$\mathrm{Tr}\left(T^{(r)}\right) = 4\,(K+1)^r. \tag{5.8}$$

It is not difficult to obtain the general result (following from the fact that only the diagonal blocks $t_{ii}^{(r)}$ have diagonal terms)

$$\mathrm{Tr}\left(T^{(r)}\right) = N\,(K+1)^r \tag{5.9}$$

For $N=(3,4)$ the particular solutions are given by (2.9) and (5.8) respectively.

Now let us consider the eigenstates of $T^{(r)}(\theta)$ for $N=4$, $r=1,2$. As compared to (3.1) we now have the fundamental state vectors

$$|1\rangle = \left|\begin{array}{c}1\\0\\0\\0\end{array}\right\rangle,\ |2\rangle = \left|\begin{array}{c}0\\1\\0\\0\end{array}\right\rangle,\ |3\rangle = \left|\begin{array}{c}0\\0\\1\\0\end{array}\right\rangle,\ |4\rangle = \left|\begin{array}{c}0\\0\\0\\1\end{array}\right\rangle \tag{5.10}$$

and, as before, we denote tensor products as

$$|ijk\ldots\rangle = |i\rangle\otimes|j\rangle\otimes|k\rangle\otimes\cdots \tag{5.11}$$

For a given n, as before, the set of states with

$$i+j+k+\cdots = n \tag{5.12}$$

will constitute the basis of the subspace S_n. For $r=1$ the situation is trivial. From (5.2)

$$T^{(1)} = t_{11}+t_{22}+t_{33}+t_{44} = (K+1)\,I_4, \tag{5.13}$$

$$T^{(1)}\,|i\rangle = (K+1)\,|i\rangle, \qquad (i=1,2,3,4). \tag{5.14}$$

For $r=2$, from (5.2) and (5.5),

$$\begin{aligned} T^{(2)} = & \left(K^2+1\right)P + 2K\left((11)\otimes(44)+(22)\otimes(33)+(33)\otimes(22)+(44)\otimes(11)\right) \\ & K\left(q+q^{-1}\right)\left((12)\otimes(43)+(13)\otimes(42)+(21)\otimes(34)+(24)\otimes(31)+(31)\otimes(24)\right. \\ & \left.+(34)\otimes(21)+(42)\otimes(13)+(43)\otimes(12)\right)+K\left(q^2+q^{-2}\right)\left((14)\otimes(41)+(41)\otimes(14)\right), \end{aligned} \tag{5.15}$$

where

$$P=\sum_{ij}(ij)\otimes(ji), \qquad (i,j=1,2,3,4)\,. \tag{5.16}$$

Implementing the definitions (5.10)-(5.12) one obtains from (5.15) for the subspaces $(S_2,\ldots,S_8)$ the following results (with $\epsilon=\pm 1$)

$$\begin{aligned}
S_2: &\quad T^{(2)}\left|11\right\rangle = \left(K^2+1\right)\left|11\right\rangle, \\
S_8: &\quad T^{(2)}\left|44\right\rangle = \left(K^2+1\right)\left|44\right\rangle, \\
S_3: &\quad T^{(2)}\left(\left|12\right\rangle+\epsilon\left|21\right\rangle\right) = \epsilon\left(K^2+1\right)\left(\left|12\right\rangle+\epsilon\left|21\right\rangle\right), \\
S_7: &\quad T^{(2)}\left(\left|43\right\rangle+\epsilon\left|34\right\rangle\right) = \epsilon\left(K^2+1\right)\left(\left|43\right\rangle+\epsilon\left|34\right\rangle\right), \\
S_4: &\quad T^{(2)}\left|22\right\rangle = \left(K^2+1\right)\left|22\right\rangle, \\
&\quad T^{(2)}\left(\left|13\right\rangle+\epsilon\left|31\right\rangle\right) = \epsilon\left(K^2+1\right)\left(\left|13\right\rangle+\epsilon\left|31\right\rangle\right), \\
S_6: &\quad T^{(2)}\left|33\right\rangle = \left(K^2+1\right)\left|33\right\rangle, \\
&\quad T^{(2)}\left(\left|42\right\rangle+\epsilon\left|24\right\rangle\right) = \epsilon\left(K^2+1\right)\left(\left|42\right\rangle+\epsilon\left|24\right\rangle\right), \qquad (5.17) \\
S_5: &\quad T^{(2)}\left(\left|14\right\rangle-\left|41\right\rangle\right) = -\left(\left(K^2-2K+1\right)+K\left(q^2+q^{-2}\right)\right)\left(\left|14\right\rangle-\left|41\right\rangle\right) \qquad (5.18) \\
&\quad T^{(2)}\left(\left|23\right\rangle-\left|32\right\rangle\right) = -\left(K^2-2K+1\right)\left(\left|23\right\rangle-\left|32\right\rangle\right) \qquad (5.19)
\end{aligned}$$

Finally, denoting

$$\left|A\right\rangle = \left(\left|14\right\rangle+\left|41\right\rangle\right), \qquad \left|B\right\rangle = \left(\left|23\right\rangle+\left|32\right\rangle\right) \tag{5.20}$$

and setting

$$T^{(2)}\left(a\left|A\right\rangle+b\left|B\right\rangle\right) = v\left(a\left|A\right\rangle+b\left|B\right\rangle\right), \tag{5.21}$$

where

$$v=K^2+1+K\cdot f \tag{5.22}$$

f being K-independent (a function $f(q)$ of q only), one obtains the constraints

$$a\left(\left(q+q^{-1}\right)^2-f\right)+b\cdot 2\left(q+q^{-1}\right)=0, \qquad a\cdot 2\left(q+q^{-1}\right)+b\left(2-f\right)=0. \tag{5.23}$$

Hence

$$f=\frac{1}{2}\left(q^2+q^{-2}+4\right)\pm\frac{1}{2}\sqrt{\left(q+q^{-1}\right)^4+12\left(q+q^{-1}\right)^2+4} \tag{5.24}$$

with corresponding K-independent values of (a,b). The sum of the eigenvalues given by (5.17)-(5.24) is

$$4\left(K+1\right)^2 \tag{5.25}$$

consistently with (5.8) for $r=2$. Our explicit results for $r=2$ not only shows how the basic properties (1), (2), (3) stated at the beginning of this section are all realized systematically but also how (3.14) is carried over, the subspaces now being paired via

$$(1,2,3,4)\rightarrow(4,3,2,1) \tag{5.26}$$

The chain Hamiltonian for any N is given by (4.1)-(4.3) with (1.4)-(1.7) giving P_0' in $\dot{\hat{R}}_0 = \dot{K}_0 P_0'$. In $\dot{K}_0$ of (4.5) now, from (1.3), $e^{\eta} + e^{-\eta} = [N-1] + 1$. For $r = 2$, one obtains for example

$$H^{(2)} = \dot{K}_0 \left(P_0' + P P_0' P\right) = \dot{K}_0 \left(\sum_{i,j=1}^{N} \left(q^{\rho_{i'} - \rho_j} + q^{\rho_i - \rho_j'} \right) (ij) \otimes (i'j') \right). \qquad (5.27)$$

For $N = 4$ this corresponds to

$$\begin{aligned} \left(\dot{K}_0\right)^{-1} H^{(2)} = {} & \left(q^{-2} + q^2\right)(11) \otimes (44) + \left(q^{-1} + q\right)(12) \otimes (43) + \left(q^{-1} + q\right)(13) \otimes (42) + 2(14) \otimes (41) \\ & \left(q^{-1} + q\right)(21) \otimes (34) + 2(22) \otimes (33) + 2(23) \otimes (32) + \left(q^{-1} + q\right)(24) \otimes (31) \\ & \left(q^{-1} + q\right)(31) \otimes (24) + 2(32) \otimes (23) + 2(33) \otimes (22) + \left(q^{-1} + q\right)(34) \otimes (21) \qquad (5.28) \\ & 2(41) \otimes (14) + \left(q^{-1} + q\right)(42) \otimes (13) + \left(q^{-1} + q\right)(43) \otimes (12) + \left(q^{-2} + q^2\right)(44) \otimes (11). \end{aligned}$$

Generalizations for $r > 2$ can be written down systematically. If the "spin" associated with the state $|i\rangle$ is denoted as σ_i then (4.1) along with structures analogous to (5.27) implies transitions (if the states of two neighboring sites have spins σ_j, $\sigma_{j'}$)

$$(\sigma_j, \sigma_{j'}) \longrightarrow (\sigma_i, \sigma_{i'}) \qquad (5.29)$$

with evident q-dependent transition amplitudes corresponding to the matrix elements of $H^{(r)}$ for order r. In particular if, for example,

$$(\sigma_1, \sigma_2, \ldots, \sigma_{N-1}, \sigma_N) = \left(\frac{N-1}{2}, \frac{N-2}{2}, \ldots, -\frac{N-2}{2}, -\frac{N-1}{2} \right) \qquad (5.30)$$

then

$$\sigma_i + \sigma_{i'} = 0 \qquad (i' = N - i + 1). \qquad (5.31)$$

For $N = 3$ (as discussed in (4.26)-(4.28))

$$(\sigma_1, \sigma_2, \sigma_3) = (1, 0, -1) \qquad (5.32)$$

and for $N = 4$

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right) \qquad (5.33)$$

and so on.

If for two adjacent sites (including circular boundary constraints) one has states $(|i\rangle, |i'\rangle)$ they can be flipped to any pair $(|j\rangle, |j'\rangle)$. Thus such a flip can propagate along the chain for any state $|j\rangle$ of the next site.

A thorough investigation of our class of models for arbitrary N is beyond the scope of the present paper. We have however indicated how the basic features studied for $N = 3$ are carried over as N increases. Such properties are conserved due to the specific structure of P_0' as defined in (1.4)-(1.7).

We just mention finally that features parallel to those discussed for $N = 3$ in (3.43)-(3.49) can be carried over starting with the multinomial expansion of

$$(x_1 + x_2 + x_3 + \cdots + x_N)^r. \qquad (5.34)$$

Dimensions of invariant subspaces are obtained entirely analogously.

6 Potential for factorizable S-matrix $(N = 3)$

As in Sec. 5 of Ref. 3 we construct the inverse Cayley transform of the YB matrix which is also the $t^{(1)}(\theta)$ matrix (2.1) and given by (2.2) for $N = 3$ for the class studied in this paper. The role of this in providing the potential for factorizable S-matrices can be found in various sources [5, 6]. As explained and emphasized in Sec. 5 of Ref. 3 an arbitrary normalization factor (denoted $\lambda^{-1}(\theta)$) of $R(\theta)$ must be introduced to start with for the inversion involved in the transform to be well-defined. The explicit inversion in the first factor of

$$-iV = (R(\theta) - \lambda(\theta) I)^{-1} (R(\theta) + \lambda(\theta) I) \tag{6.1}$$

will display admissible choices of $\lambda(\theta)$. Defining

$$X(R(\theta) - \lambda(\theta) I) = I, \qquad -iV = X\left(X^{-1} + 2\lambda(\theta) I\right) = I + 2\lambda(\theta) X, \tag{6.2}$$

for $N = 3$, (2.1) leads to (suppressing the argument θ in notation below)

$$X \begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K-\lambda & 0 & q^{1/2}K & 0 & 1+qK & 0 & 0 \\ 0 & 1 & 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1/2}K & 0 & 1+K-\lambda & 0 & q^{1/2}K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda & 0 & 1 & 0 \\ 0 & 0 & 1+q^{-1}K & 0 & q^{-1/2}K & 0 & K-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\lambda \end{vmatrix} = I, \tag{6.3}$$

Only the non-zero elements of X will be given below. One obtains easily

$$(X_{11}, X_{99}) = (1-\lambda)^{-1}, \qquad (X_{22}, X_{44}, X_{66}, X_{88}) = \frac{\lambda}{(1-\lambda^2)}, \qquad (X_{24}, X_{42}, X_{68}, X_{86}) = \frac{1}{(1-\lambda^2)}. \tag{6.4}$$

These already show $\lambda \neq \pm 1$. For $i = (3, 5, 7)$ one obtains the equations

$$(-X_{i3}\lambda + X_{i7}) + q^{-1/2} K Z_i = \delta_{i3}, \qquad X_{i5}(1-\lambda) + K Z_i = \delta_{i5}, \qquad (X_{i3} - \lambda X_{i7}) + K q^{1/2} Z_i = \delta_{i7}, \tag{6.5}$$

where

$$Z_i \equiv q^{1/2} X_{i3} + X_{i5} + q^{-1/2} X_{i7}. \tag{6.6}$$

The solutions for $i = (3, 5, 7)$ are respectively the following ones. For $i = 3$, (X_{33}, X_{35}, X_{37}) are given by

$$X_{33} = \frac{\lambda}{1-\lambda^2} + q^{-1/2}\left(\frac{q+\lambda}{1+\lambda}\right) X_{35}, \tag{6.7}$$

$$X_{37} = \frac{1}{1-\lambda^2} + q^{-1/2}\left(\frac{1+q\lambda}{1+\lambda}\right) X_{35}, \tag{6.8}$$

$$Z_3 = \frac{q^{-1/2}(1+q\lambda)}{1-\lambda^2} + \left(\frac{q+q^{-1}+1+3\lambda}{1+\lambda}\right) X_{35}, \tag{6.9}$$

$$X_{35}(1-\lambda) + K Z_3 = 0. \tag{6.10}$$

The K-dependence is now explicit. The case $i = 3, 7$ are related though the exchange of indices and inversion of q, namely

$$(3, 7; q) \rightleftharpoons (7, 3; q^{-1}) \tag{6.11}$$

For $i = 5$

$$X_{53}\left(1+q\lambda\right) = X_{57}\left(q+\lambda\right), \tag{6.12}$$

$$X_{53} = -\frac{q^{-1/2}\left(q+\lambda\right)}{1-\lambda^2} + \frac{\left(q+\lambda\right)q^{-1/2}}{1+\lambda}X_{55}, \tag{6.13}$$

$$X_{57} = -\frac{q^{-1/2}\left(1+q\lambda\right)}{1-\lambda^2} + \frac{\left(1+q\lambda\right)q^{-1/2}}{1+\lambda}X_{55}, \tag{6.14}$$

$$Z_5 = -\frac{q+q^{-1}+2\lambda}{1-\lambda^2} + \frac{3\lambda+1+q+q^{-1}}{1+\lambda}X_{55}, \tag{6.15}$$

$$X_{55}\left(1-\lambda\right) + KZ_5 = 1. \tag{6.16}$$

Now (6.19), (6.20) gives directly X_{55}. Next (6.17), (6.18) give X_{53}, X_{57}. Finally, we obtain

$$X = \begin{vmatrix}
\frac{1}{1-\lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\lambda}{1-\lambda^2} & 0 & \frac{1}{1-\lambda^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathsf{A} & 0 & \mathsf{B} & 0 & \mathsf{C} & 0 & 0 \\
0 & \frac{1}{1-\lambda^2} & 0 & \frac{\lambda}{1-\lambda^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathsf{D} & 0 & \mathsf{E} & 0 & \mathsf{B} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\lambda}{1-\lambda^2} & 0 & \frac{1}{1-\lambda^2} & 0 \\
0 & 0 & \mathsf{F} & 0 & \mathsf{D} & 0 & \mathsf{A} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{1-\lambda^2} & 0 & \frac{\lambda}{1-\lambda^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{1-\lambda}
\end{vmatrix}, \tag{6.17}$$

where

$$\mathsf{A} = \frac{\left(\lambda^2-\lambda-2K\lambda+K\right)q}{\left(1-\lambda\right)\left(q\lambda^2-3qK\lambda-q^2K-K-qK-q\right)} = X_{33} = X_{77},$$

$$\mathsf{B} = \frac{\left(q\lambda+1\right)q^{1/2}K}{\left(1-\lambda\right)\left(q\lambda^2-3qK\lambda-q^2K-K-qK-q\right)} = X_{35} = X_{57},$$

$$\mathsf{C} = \frac{\left(\lambda+qK\lambda-qK-1-K\right)q}{\left(1-\lambda\right)\left(q\lambda^2-3qK\lambda-q^2K-K-qK-q\right)} = X_{37},$$

$$\mathsf{D} = \frac{K\left(\lambda+q\right)q^{1/2}}{\left(1-\lambda\right)\left(q\lambda^2-3qK\lambda-q^2K-K-qK-q\right)} = X_{53} = X_{75},$$

$$\mathsf{E} = \frac{q\lambda^2-2Kq\lambda-q^2K-K-q}{\left(1-\lambda\right)\left(q\lambda^2-3qK\lambda-q^2K-K-qK-q\right)} = X_{55},$$

$$\mathsf{F} = \frac{-q-qK+q\lambda-K+K\lambda}{\left(1-\lambda\right)\left(q\lambda^2-3qK\lambda-q^2K-K-qK-q\right)} = X_{73}, \tag{6.18}$$

where $\lambda \neq \pm 1, \frac{1}{2}\left[3K \pm \sqrt{9K^2+4\left(q+1+q^{-1}\right)K+4}\right]$.

From X, V is obtained as indicated in (6.2). Expressing it as

$$V = \sum_{ab,cd} V_{(ab,cd)}\left(ab\right)\otimes\left(cd\right) \tag{6.19}$$

The corresponding fermionic Lagrangian should be

$$\mathcal{L} = \int dx\left(i\overline{\psi}_a\gamma_\nu\partial_\nu\psi_a - g\left(\overline{\psi}_a\gamma_\nu\psi_c\right)V_{ab,cd}\left(\overline{\psi}_b\gamma_\nu\psi_d\right)\right), \tag{6.20}$$

The scalar Lagrangian can be obtained analogously. Such Lagrangians correspond to S-matrices factorizable into two particles scattering independently of the chosen order of the latter ones.

7 Discussion

In Ref. 3 and in the present paper we have studied two different classes of statistical models. Certain aspects of the respective transfer matrices are strikingly contrasted. Such a major difference is in the number of parameters. The first model is indeed multiparameter. One has $\frac{1}{2}(N+3)(N-1)$ free parameters ($N = 3, 4, \ldots$). Here the only parameter is q appearing in the braid matrix given by (1.2)-(1.7) and in $K(\theta)$ as defined by (1.17)-(1.20). The structures of the eigenvalues of the respective transfer matrices are also quite different. In Ref. 3 we obtained single exponentials as eigenvalues, the exponent being a sum of the free parameters multiplied by θ. Here we have r-th order polynomials in $K(\theta)$ for the eigenvalues of $T^{(r)}(\theta)$. There are other differences. But analogies and common features are also remarkable:

(a) In both case $\mathrm{Tr}\left(T^{(r)}(\theta)\right)$ is obtained quite simply for all r (though the structures are different). In (6.1) of Ref. 3 we obtained (for $N = 2p-1$)

$$\mathrm{Tr}\left(T^{(r)}(\theta)\right) = 2\left(e^{rm_{11}^{(+)}\theta} + e^{rm_{22}^{(+)}\theta} + \cdots + e^{rm_{p-1,p-1}^{(+)}\theta}\right) + 1 \tag{7.1}$$

the $m_{ii}^{(+)}$ being a subset of the free parameters. Here (for $N = 3, 4, \ldots$) the corresponding result (5.9) is

$$\mathrm{Tr}\left(T^{(r)}(\theta)\right) = N\left(K(\theta)+1\right)^r, \tag{7.2}$$

where $K(\theta)$ is given by (1.3) and (1.17).

(b) In both cases the N^r dimensional base space of $T^{(r)}(\theta)$ breaks up into closed subspaces of lower dimensions. The definitions of these subspaces have some differences however. The relevant definitions in Ref. 3 should be compared to (3.2)-(3.6) here and their generalization in Sec. 5.

(c) In each subspace S_n the circular permutation of states labels as formulated in (3.9)-(3.19) leads to a further reduction of dimension in constructing eigenstates by splitting S_n again into subsets corresponding to the eigenstates of the operator (CP) of circular permutations. This involves a crucial role of the roots of unity, ($\omega^r = 1$ for $T^{(r)}(\theta)$) in the construction of eigenstates. The role of roots of unity was also crucial in Ref. 3 though they were implemented in a slightly different fashion (corresponding to the difference in labeling states).

In both cases the "two-step reduction" (via (b) and (c)) in the effective dimension of the basis in construction of eigenstates has been emphasized (see the formulation of Sec. 3). The exponential increase in dimension with r $\left(e^{(\ln N)r}\right)$ is replaced in actual construction by a relatively moderate polynomial one. Thus for $N = 3$ and $r = 4$ we have to solve here at most a set of 5 simultaneous linear equations (App. A) though now N^r is $3^4 = 81$. This reduction of the problem to a relatively low number of linear equations should be contrasted to the implementation of algebraic Bethe ansatz [6, 7, 8]. For the latter one has to solve complex nonlinear equations whose number increases along with N.

In the preceding sections (particularly in Sec. 3 for $N = 3$ and in sec. 5 for $N > 3$), we have formulated carefully the crucial properties, basic features of models corresponding to the braid matrices presented in Ref. 2. Exploiting such properties we have constructed eigenstates and eigenvalues of $T^{(r)}(\theta)$ for $N = 3$, $r = (1, 2, 3, 4)$ (App. A). Certain related features for all r have been formulated at the end of Sec. 3. Chain Hamiltonians and potentials for factorizable S-matrices have been studied (sections 4 and 6).

Further explorations in several directions are evidently desirable. Reflection equations [9, 10] and correlation functions [11, 12] should be studied. More basically one may try to elucidate the relevance of the star-triangle relations [4] encoded in our class of braid matrices to specific contexts. We hope to undertake such studies elsewhere.

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Appendix A
Eigenstates and Eigenvalues of $T^{(r)}(\theta)$ for $r = 1, 2, 3, 4$ $(N = 3)$

We start by noting the following points:

(1) For each case the subscript n of S_n denotes the sum of the state labels (see discussion from (3.3) to (3.6)).

(2) For each r we present results only upto S_{2r}. The remaining subspaces $(S_{2r+1}, \ldots, S_{3r})$ are then obtained implementing (3.14), (3.15), (3.16).

(3) For different subspaces we often repeat the some notations for states. Since $T^{(r)}$ does not couple such spaces no confusion is likely.

(4) The notation K and ω correspond to (1.17) and (3.11) respectively. (CP) denotes circular permutations.

A.1 $r = 1$

The three states directly furnish the spectrum, each being a 1-dimensional subspace

$$T^{(1)}(\theta)\left(|1\rangle, |2\rangle, |3\rangle\right) = (1 + K)\left(|1\rangle, |2\rangle, |3\rangle\right) \tag{A.1}$$
$$\mathrm{Tr}\left(T^{(1)}(\theta)\right) = 3(1 + K) \tag{A.2}$$

A.2 $r = 2$

The (CP) eigenstates constructed as in (3.10) to (3.13), with $\omega^2 = 1$, give

$$\begin{aligned} S_2 :& \quad A_1 = |11\rangle \\ S_3 :& \quad A_{\pm 1} = |12\rangle \pm |21\rangle \\ S_4 :& \quad A_1 = |22\rangle, \qquad B_{\pm 1} = |13\rangle \pm |31\rangle \end{aligned} \tag{A.3}$$

One obtains (K being $K(\theta)$)

$$S_2 : \; T^{(2)}(\theta)\, A_1 = \left(K^2 + 1\right) A_1 \tag{A.4}$$
$$S_3 : \; T^{(2)}(\theta)\, A_{\pm 1} = \pm\left(K^2 + 1\right) A_{\pm 1} \tag{A.5}$$
$$\begin{aligned} S_4 : \; & T^{(2)}(\theta)\, B_{-1} = -\left(K^2 + \left(q + q^{-1} - 2\right) K + 1\right) B_{-1} \\ & T^{(2)}(\theta)\left(B_1 - \left(q^{1/2} + q^{-1/2}\right) A_1\right) = \left(K^2 + 1\right)\left(B_1 - \left(q^{1/2} + q^{-1/2}\right) A_1\right) \\ & T^{(2)}(\theta)\left(\left(q^{1/2} + q^{-1/2}\right) B_1 + 2A_1\right) = \left(K^2 + \left(q + q^{-1} + 4\right) K + 1\right)\left(\left(q^{1/2} + q^{-1/2}\right) B_1 + 2A_1\right). \end{aligned} \tag{A.6}$$

Also

$$(S_5, S_6) \rightleftharpoons (S_3, S_2) \tag{A.7}$$

according to (3.14)-(3.16). Summing over all subspaces $(S_2, \ldots, S_6)$,

$$\mathrm{Tr}\left(T^{(1)}(\theta)\right) = 3\,(1 + K)^2 \tag{A.8}$$

consistently with (2.9). We have not uniformly normalized the states. Thus $\langle A_1 | A_1 \rangle = 1$ and $\langle B_{\pm 1} | B_{\pm 1} \rangle = 2$. This is crucial to the orthogonality

$$\left\langle B_1 - \left(q^{1/2} + q^{-1/2}\right) A_1 \right| \left(q^{1/2} + q^{-1/2}\right) B_1 + 2A_1 \Big\rangle = 0. \tag{A.9}$$

This point displayed here for this simple case will not be repeated in cases to follow.

A.3 $\quad r=3$

Here

$$\omega=\left(1,e^{i\frac{2\pi}{3}},e^{i\frac{2\pi}{3}\cdot 2}\right) \tag{A.10}$$

and (CP) eigenstates for (S_3,S_4,S_5,S_6) are

$$S_3:\ A_1=|111\rangle \tag{A.11}$$
$$S_4:\ A_\omega=|112\rangle+\omega\,|211\rangle+\omega^2\,|121\rangle \tag{A.12}$$
$$S_5:\ A_\omega=|113\rangle+\omega\,|311\rangle+\omega^2\,|131\rangle\,,\ B_\omega=|122\rangle+\omega\,|212\rangle+\omega^2\,|221\rangle \tag{A.13}$$
$$S_6:\ A_1=|222\rangle\,,\ B_\omega=|123\rangle+\omega\,|312\rangle+\omega^2\,|231\rangle\,,$$
$$C_\omega=|321\rangle+\omega\,|132\rangle+\omega^2\,|213\rangle=(1\rightleftharpoons 3)\,B_\omega. \tag{A.14}$$

Also

$$(1\rightleftharpoons 3)\,(S_3,S_4,S_5)=(S_9,S_8,S_7) \tag{A.15}$$

The $T^{(3)}(\theta)$ eigenstates are now obtained as follows:

$$S_3:\qquad T^{(3)}(\theta)\,A_1=\left(K^3+1\right)A_1 \tag{A.16}$$
$$S_4:\qquad T^{(3)}(\theta)\,A_\omega=\left(K^3\omega+\omega^2\right)A_\omega \tag{A.17}$$
$$S_5:\qquad T^{(3)}(\theta)\,(aA_\omega+bB_\omega)=v\,(aA_\omega+bB_\omega) \tag{A.18}$$

Solutions:

$$\textbf{(1)}\quad (a,b)=\left(q^{1/2}+\omega q^{-1/2},1\right),$$
$$v=\omega^2K^3+\left(q+q^{-1}\right)\left(\omega^2K^2+\omega K\right)+\left(1+\omega+\omega^2\right)\left(K^2+K\right)+\omega \tag{A.19}$$
$$\textbf{(2)}\quad (a,b)=\left(1,-\left(q^{1/2}+\omega^2q^{-1/2}\right)\right),\qquad v=K^3\omega^2+\omega. \tag{A.20}$$

◇ <u>S_6</u>**:** For $\omega=e^{\pm i\frac{2\pi}{3}}$, A_1 is decoupled. Set

$$T^{(3)}(\theta)\,(bB_\omega+cC_\omega)=v\,(bB_\omega+cC_\omega)\,. \tag{A.21}$$

Solutions:

$$\textbf{(1)}\quad (b,c)=(q,-1)\,,\qquad v=K^3\omega^2+\omega \tag{A.22}$$
$$\textbf{(2)}\quad (b,c)=(1,q)\,,\qquad v=K^3\omega^2+\omega+\left(q+q^{-1}\right)\left(K^2\omega^2+K\omega\right). \tag{A.23}$$

For the values of ω ($\neq 1$), with $\omega+\omega^2=-1$ the sum of eigenvalues

$$\sum v=-2\left(K^3+1\right)-\left(q+q^{-1}\right)\left(K^2+K\right). \tag{A.24}$$

For $\omega=1$, $T^{(3)}$ couples (A_1,B_1,C_1). Set

$$T^{(3)}\,(\alpha A_1+\beta B_1+\gamma C_1)=v_1\,(\alpha A_1+\beta B_1+\gamma C_1)\,. \tag{A.25}$$

Solutions:

$$\textbf{(1)}\quad (\alpha,\beta,\gamma)=\left(0,q^{1/2},-q^{-1/2}\right),\qquad v_1=(K+1)\left(K^2-K+1\right) \tag{A.26}$$
$$\textbf{(2)}\quad (\alpha,\beta,\gamma)=\left(-\left(q+q^{-1}\right),q^{-1/2},q^{1/2}\right),\qquad v_1=(K+1)\left(K^2-K+1\right) \tag{A.27}$$
$$\textbf{(3)}\quad (\alpha,\beta,\gamma)=\left(3,q^{-1/2},q^{1/2}\right),\qquad v_1=(K+1)\left(\left(K^2-K+1\right)+K\left(q+q^{-1}+3\right)\right) \tag{A.28}$$

Concerning orthogonality note that

$$\langle A_1|A_1\rangle = 1, \qquad \langle B_1|B_1\rangle = \langle C_1|C_1\rangle = 3. \tag{A.29}$$

The sum of the eigenvalues over S_6 is

$$\left(\sum v\right)_{S_6} = K^3 + 1 + 3K\left(K+1\right). \tag{A.30}$$

The results for (S_7, S_8, S_9) are obtained, as usual, directly from those of (S_5, S_4, S_3) respectively.

Summing over all the subspaces $(S_3, \ldots, S_9)$ one obtains (all explicit q-dependence canceling consistently with (2.9))

$$\begin{aligned} Tr\left(T^{(3)}\left(\theta\right)\right) &= \left(K^3+1\right) + \left(K^3+1\right) + 3\left(K^2+K\right) + 3\left(K^2+K\right) + \left(K^3+1\right) + 3\left(K^2+K\right) \\ &= 3\left(K+1\right)^3. \end{aligned} \tag{A.31}$$

A.4 $r = 4$

Here

$$\omega = \left(1, e^{i\frac{2\pi}{4}}, e^{i\frac{2\pi}{4}\cdot 2}, e^{i\frac{2\pi}{4}\cdot 3}\right) = \left(1, i, -1, -i\right). \tag{A.32}$$

Of the invariant subspaces we consider $(S_4, S_5, S_6, S_7, S_8)$. One obtains the results of the remaining ones via $(1,2,3;q) \leftrightarrow (3,2,1;q^{-1})$ as

$$\left(S_9, S_{10}, S_{11}, S_{12}\right) \rightleftharpoons \left(S_7, S_6, S_5, S_4\right). \tag{A.33}$$

For brevity and simplicity, we will recapitulate our results in the following tables:

Table 1: *CP eigenstates for* $r = 4$

Subspace	CP Eigenstates	Dimension
S_4	$\|1111\rangle$	1
S_5	$A_\omega = \|1112\rangle + \omega\,\|2111\rangle + \omega^2\,\|1211\rangle + \omega^3\,\|1121\rangle$	4
S_6	$A_{\pm 1} = \|1212\rangle \pm \|2121\rangle$ $B_\omega = \|1113\rangle + \omega\,\|3111\rangle + \omega^2\,\|1311\rangle + \omega^3\,\|1131\rangle$ $C_\omega = \|1122\rangle + \omega\,\|2112\rangle + \omega^2\,\|2211\rangle + \omega^3\,\|1221\rangle$	10
S_7	$A_\omega = \|1222\rangle + \omega\,\|2122\rangle + \omega^2\,\|2212\rangle + \omega^3\,\|2221\rangle$ $B_\omega = \|1123\rangle + \omega\,\|3112\rangle + \omega^2\,\|2311\rangle + \omega^3\,\|1231\rangle$ $C_\omega = \|1132\rangle + \omega\,\|2113\rangle + \omega^2\,\|3211\rangle + \omega^3\,\|1321\rangle$ $D_\omega = \|1213\rangle + \omega\,\|3121\rangle + \omega^2\,\|1312\rangle + \omega^3\,\|2131\rangle$	16
S_8	$A_1 = \|2222\rangle$ $B_{\pm 1} = \|1313\rangle \pm \omega\,\|3131\rangle$ $C_\omega = \|1133\rangle + \omega\,\|3113\rangle + \omega^2\,\|3311\rangle + \omega^3\,\|1331\rangle$ $D_\omega = \|1223\rangle + \omega\,\|3122\rangle + \omega^2\,\|2312\rangle + \omega^3\,\|2231\rangle$ $E_\omega = \|3221\rangle + \omega\,\|1322\rangle + \omega^2\,\|2132\rangle + \omega^3\,\|2213\rangle$ $F_\omega = \|1232\rangle + \omega\,\|2123\rangle + \omega^2\,\|3212\rangle + \omega^3\,\|2321\rangle$	19

Table 2: *Eigenstates and eigenvalues for $r = 4$*

	Eigenvalues	Eigenstates
$S_4:$	K^4+1	$\|1111\rangle$
$S_5:$	$\omega^3 K^4+\omega$	A_ω
$S_6:$	$\pm\left(K^4+1\right)$	$A_{\pm 1}$
	$\omega^3 K^4+\omega$	$B_\omega - \frac{q+\omega^3}{\sqrt{q}} C_\omega$
	$\omega^3\left[K^4+(q+1+\omega+\omega^3+q^{-1})K^3+\right.$ $\omega^3(q+1+\omega+\omega^3+q^{-1})K^2+$ $\left.\omega^2(q+1+\omega+\omega^3+q^{-1})K+\omega^2\right]$	$B_\omega + \frac{\sqrt{q}}{q+\omega} C_\omega$
$S_7:$	$\omega^3 K^4+\omega$	$A_\omega - \omega\sqrt{q} B_\omega - \frac{1}{\sqrt{q}} C_\omega,$ $-\left(q^3+\omega^2\right) A_\omega + q\sqrt{q}\left(q+1+q^{-1}\right) D_\omega +$ $\omega\sqrt{q}\left(\omega^2-q^2-q\right) B_\omega + \sqrt{q}\left(q^2-\omega^2 q-\omega^2\right) C_\omega$
	$\omega^3 K^4+\omega+\left(\omega^3 K^3+\omega K\right)\left(q+1+\omega+\omega^3+q^{-1}\right)$ $+K^2\omega^2\left(q+1+\omega+\omega^3+q^{-1}\right)$	$\frac{1+\omega}{\sqrt{q}} A_\omega + \frac{\omega^2}{q} B_\omega + C_\omega + \omega\left(\frac{q+\omega}{q}\right) D_\omega$
	$\omega^3 K^4+\omega+\left(\omega^3 K^3+\omega K\right)\left(q+1-\omega-\omega^3+q^{-1}\right)$ $+K^2\omega^2\left(q+1-\omega-\omega^3+q^{-1}\right)$	$\frac{1-\omega}{\sqrt{q}} A_\omega - \frac{\omega^2}{q} B_\omega + C_\omega - \omega\left(\frac{q-\omega}{q}\right) D_\omega$
$S_8:$	K^4+1	$F_1,\ 2A_1 + C_1 - \sqrt{q} D_1 - \frac{1}{\sqrt{q}} E_1,$ $2\sqrt{q}\left(q^2+q-2+q^{-1}+q^{-2}\right) A_1 +$ $2\sqrt{q}\left(q+2+q^{-1}\right) B_1 - \sqrt{q}\left(q+4+q^{-1}\right) C_1 +$ $\left(q^2+2q-3-2q^{-1}-3\right) D_1$ $-\left(2q^2+3q-2-q^{-1}\right) E_1$
	$-K^4-1$	$F_{-1},$ $B_{-1} + \left(\frac{q^2-1}{2q}\right) C_{-1} - \left(\frac{q^2+1}{2\sqrt{q}}\right) D_{-1} + \left(\frac{q^2+1}{2q\sqrt{q}}\right) E_{-1}$
	$\mp i K^4 \pm i$	$F_{\pm i},\ \sqrt{q} C_{\pm i} - D_{\pm i} + E_{\pm i}$
	$\mp i\left[K^4+\left(q+1+q^{-1}\right)K^3 \pm 3iK^2-\right.$ $\left.\left(q+1+q^{-1}\right)K-1\right]$	$-\left(\frac{q+1}{\sqrt{q}}\right) C_{\pm i} - D_{\pm i} + E_{\pm i}$
	$\mp i\left[K^4+\left(q+1+q^{-1}\right)K^3 \mp i\left(q+1+q^{-1}\right)K^2-\right.$ $\left.\left(q+1+q^{-1}\right)K-1\right]$	$C_{\pm i} + \frac{1+2q}{(q-1)\sqrt{q}} D_{\pm i} + \frac{(q+2)\sqrt{q}}{q-1} E_{\pm i}$

$K^4 + (q+3+q^{-1})\,K^3 + (q+3+q^{-1})\,K^2 + (q+3+q^{-1})\,K + 1$	$-4A_1 + 2B_1 + C_1 - \frac{1}{(q+1)\sqrt{q}} D_1 - \frac{q\sqrt{q}}{q+1} E_1$
$-\left[K^4 + (q-1+q^{-1})\,K^3 - (q-1+q^{-1})\,K^2 + (q-1+q^{-1})\,K + 1\right]$	$-\frac{2(q-1)}{q+1} B_{-1} + C_{-1} - \frac{1}{(q+1)\sqrt{q}} D_{-1} - \frac{q\sqrt{q}}{q+1} E_{-1}$

$\star$ There exist also four others eigenvectors $aA_1+bB_1+cC_1+dD_1+eE_1$, $\alpha A_1+\beta B_1+\gamma C_1+\delta D_1+\eta E_1$, $b'B_{-1}+c'C_{-1}+d'D_{-1}+e'E_{-1}$, $\beta' B_{-1}+\gamma' C_{-1}+\delta' D_{-1}+\eta' E_{-1}$ associated respectively to the eigenvalues v_1, v_2, v_1' and v_2', which have complicated forms (these results have been obtained by using a Maple program):

$$\begin{aligned}
v_1 &= \frac{1}{2}\Big(2K^4 + 3K^3\left(q+1+q^{-1}\right) + K^2\left(q^2+2q+13+2q^{-1}+q^{-2}\right) + 3K\left(q+1+q^{-1}\right) + 2 + \\
&\quad K\left(K^2 + \left(q+1+q^{-1}\right)K + 1\right)\sqrt{q^2+2q+43+2q^{-1}+q^{-2}}\Big), \\
v_2 &= \frac{1}{2}\Big(2K^4 + 3K^3\left(q+1+q^{-1}\right) + K^2\left(q^2+2q+13+2q^{-1}+q^{-2}\right) + 3K\left(q+1+q^{-1}\right) + 2 - \\
&\quad K\left(K^2 + \left(q+1+q^{-1}\right)K + 1\right)\sqrt{q^2+2q+43+2q^{-1}+q^{-2}}\Big), \\
v_1' &= \frac{1}{2}\Big(-2K^4 - 3K^3\left(q+1+q^{-1}\right) - K^2\left(q^2+2q+1+2q^{-1}+q^{-2}\right) - 3K\left(q+1+q^{-1}\right) - 2 + \\
&\quad K\left(K^2 + \left(q+1+q^{-1}\right)K + 1\right)\sqrt{q^2+2q-5+2q^{-1}+q^{-2}}\Big), \\
v_2' &= \frac{1}{2}\Big(-2K^4 - 3K^3\left(q+1+q^{-1}\right) - K^2\left(q^2+2q+1+2q^{-1}+q^{-2}\right) - 3K\left(q+1+q^{-1}\right) - 2 - \\
&\quad K\left(K^2 + \left(q+1+q^{-1}\right)K + 1\right)\sqrt{q^2+2q-5+2q^{-1}+q^{-2}}\Big).
\end{aligned} \tag{A.34}$$

The sums of the eigenvalues over S_4, S_5, S_6, S_7 and S_8 are respectively

$$\begin{aligned}
\sum_{S_4} v &= K^4 + 1, \\
\sum_{S_5} v &= 0, \\
\sum_{S_6} v &= 4K^3 + 4K, \\
\sum_{S_7} v &= 0, \\
\sum_{S_8} v &= K^4 + 4K^3 + 18K^2 + 4K + 1.
\end{aligned} \tag{A.35}$$

The results for $(S_9, S_{10}, S_{11}, S_{12})$ are obtained directly from those of (S_7, S_6, S_5, S_4) respectively. Summing over all subspaces $(S_4, \ldots, S_{12})$ one obtains

$$\mathrm{Tr}\left(T^{(4)}(\theta)\right) = 2 \times \left(K^4+1\right) + 2 \times \left(4K^3+4K\right) + 1 \times \left(K^4+4K^3+18K^2+4K+1\right) = 3\left(K+1\right)^4. \tag{A.36}$$

Appendix B

$\hat{R}tt$-Algebra

We present below, for $N = 3$, the constraints on the blocks $t_{ij}(\theta)$ of the transfer matrix following from

$$\hat{R}(\theta - \theta')\, t(\theta) \otimes t(\theta') = t(\theta') \otimes (\theta)\, \hat{R}(\theta - \theta')\,. \tag{B.1}$$

We use below the notations

$$(t(\theta), t(\theta'), K(\theta - \theta')) \equiv (t, t', K'') \tag{B.2}$$

In terms of P_0' defined by (1.4) with (i, j) and (ρ_i, ρ_j) corresponding to $N = 3$ and $K(\theta)$ of (1.17)-(1.18), (B.1) now is (maintaining the notation P_0' unrelated to θ, θ')

$$(I + K'' P_0')(t \otimes t') = (t' \otimes t)(I + K'' P_0')\,, \tag{B.3}$$

where

$$\begin{aligned} P_0' = {}& q^{-1}(11) \otimes (33) + q^{-1/2}(12) \otimes (32) + (13) \otimes (31) + q^{-1/2}(21) \otimes (23) + (22) \otimes (22) \\ &+ q^{1/2}(23) \otimes (21) + (31) \otimes (13) + q^{1/2}(32) \otimes (12) + q(33) \otimes (11) \end{aligned} \tag{B.4}$$

This leads to a set of 36 relations independent of K'', namely

$$t_{ij} t'_{kl} = t'_{ij} t_{kl} \tag{B.5}$$

where for $(ij) = (11), (12), (13)$ respectively

$$(kl) = (11, 12, 21, 22),\ (11, 13, 21, 23),\ (12, 13, 22, 23) \tag{B.6}$$

and similarly for $(ij) = (21), (22), (23)$

$$(kl) = (11, 12, 31, 32),\ (11, 13, 31, 33),\ (12, 13, 32, 33) \tag{B.7}$$

and for $(ij) = (31), (32), (33)$

$$(kl) = (21, 22, 31, 32),\ (21, 23, 31, 33),\ (22, 23, 32, 33) \tag{B.8}$$

To present the K'' dependent constraints we first define

$$\begin{aligned} X_1 &= q^{-1/2} t_{11} t'_{31} + t_{21} t'_{21} + q^{1/2} t_{31} t'_{11}, & X_2 &= q^{-1/2} t_{11} t'_{32} + t_{21} t'_{22} + q^{1/2} t_{31} t'_{12}, \\ X_3 &= q^{-1/2} t_{11} t'_{33} + t_{21} t'_{23} + q^{1/2} t_{31} t'_{13}, & X_4 &= q^{-1/2} t_{12} t'_{31} + t_{22} t'_{21} + q^{1/2} t_{32} t'_{11}, \\ X_5 &= q^{-1/2} t_{12} t'_{32} + t_{22} t'_{22} + q^{1/2} t_{32} t'_{12}, & X_6 &= q^{-1/2} t_{12} t'_{33} + t_{22} t'_{23} + q^{1/2} t_{32} t'_{13}, \\ X_7 &= q^{-1/2} t_{13} t'_{31} + t_{23} t'_{21} + q^{1/2} t_{33} t'_{11}, & X_8 &= q^{-1/2} t_{13} t'_{32} + t_{23} t'_{22} + q^{1/2} t_{33} t'_{12}, \\ X_9 &= q^{-1/2} t_{13} t'_{33} + t_{23} t'_{23} + q^{1/2} t_{33} t'_{13} \end{aligned} \tag{B.9}$$

and a set

$$(Y_1, Y_2, \ldots, Y_9) \tag{B.10}$$

which is obtained by transposing the indices of each term on the right of (B.9) and also the order of (θ, θ'). Thus

$$Y_1 = q^{-1/2} t'_{11} t_{13} + t'_{12} t_{12} + q^{1/2} t'_{13} t_{11} \tag{B.11}$$

and so on. The constraints involving K'' only through X_i are the following ones

$$\begin{aligned}
q^{1/2}\left(t_{11}t'_{31}-t'_{11}t_{31}\right)&=\left(t_{21}t'_{21}-t'_{21}t_{21}\right)=q^{-1/2}\left(t_{31}t'_{11}-t'_{31}t_{11}\right)=-K''X_1,\\
q^{1/2}\left(t_{11}t'_{32}-t'_{11}t_{32}\right)&=\left(t_{21}t'_{22}-t'_{21}t_{22}\right)=q^{-1/2}\left(t_{31}t'_{12}-t'_{31}t_{12}\right)=-K''X_2,\\
q^{1/2}\left(t_{12}t'_{31}-t'_{12}t_{31}\right)&=\left(t_{22}t'_{21}-t'_{22}t_{21}\right)=q^{-1/2}\left(t_{32}t'_{11}-t'_{32}t_{11}\right)=-K''X_4,\\
q^{1/2}\left(t_{12}t'_{33}-t'_{12}t_{33}\right)&=\left(t_{22}t'_{23}-t'_{22}t_{23}\right)=q^{-1/2}\left(t_{32}t'_{13}-t'_{32}t_{13}\right)=-K''X_6,\\
q^{1/2}\left(t_{13}t'_{32}-t'_{13}t_{32}\right)&=\left(t_{23}t'_{22}-t'_{23}t_{22}\right)=q^{-1/2}\left(t_{33}t'_{12}-t'_{33}t_{12}\right)=-K''X_8,\\
q^{1/2}\left(t_{13}t'_{33}-t'_{13}t_{33}\right)&=\left(t_{23}t'_{23}-t'_{23}t_{23}\right)=q^{-1/2}\left(t_{33}t'_{13}-t'_{33}t_{13}\right)=-K''X_9.
\end{aligned} \tag{B.12}$$

There are six corresponding sets involving K'' only through Y_i. As for (B.11) they are obtained by transposing indices in the first three terms of each equation of (B.12) and changing the sign before K''. Thus

$$q^{1/2}\left(t_{11}t'_{13}-t'_{11}t_{13}\right)=\left(t_{12}t'_{12}-t'_{12}t_{12}\right)=q^{-1/2}\left(t_{13}t'_{11}-t'_{13}t_{11}\right)=K''Y_1 \tag{B.13}$$

and so on.

Finally there is a set involving K'' through both X_i and Y_i

$$\begin{aligned}
\left(t_{11}t'_{33}-t'_{11}t_{33}\right)&=-K''q^{-1/2}\left(X_3-Y'_3\right),\\
\left(t_{12}t'_{32}-t'_{12}t_{32}\right)&=-K''\left(q^{-1/2}X_5-Y'_3\right),\\
\left(t_{13}t'_{31}-t'_{13}t_{31}\right)&=-K''\left(q^{-1/2}X_7-q^{1/2}Y'_3\right),\\
\left(t_{21}t'_{23}-t'_{21}t_{23}\right)&=-K''\left(X_3-q^{-1/2}Y'_5\right),\\
\left(t_{22}t'_{22}-t'_{22}t_{22}\right)&=-K''\left(X_5-Y'_5\right),\\
\left(t_{23}t'_{21}-t'_{23}t_{21}\right)&=-K''\left(X_7-q^{1/2}Y'_5\right),\\
\left(t_{31}t'_{13}-t'_{31}t_{13}\right)&=-K''\left(q^{1/2}X_3-q^{-1/2}Y'_7\right),\\
\left(t_{32}t'_{12}-t'_{32}t_{12}\right)&=-K''\left(q^{1/2}X_5-Y'_7\right),\\
\left(t_{33}t'_{11}-t'_{33}t_{11}\right)&=-K''q^{1/2}\left(X_7-Y'_7\right).
\end{aligned} \tag{B.14}$$

An alternative approach to the $\hat{R}tt$ relations is via the diagonalization of P'_0. The diagonalizer is given in [13]. Such an approach was presented for our multiparameter (”nested-sequence”) class in App. C of Ref. 3.

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